An Example of Use of Category theory for AHL nets

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Outline

- Motivations
- Category
  - **PetriG, AHL** Category
- Functor
  - Semantics Functor (**AHL** $\Rightarrow$ **PetriG**)
  - Properties
- (Co)Products
  - Composition of nets in **PetriG** Category
- Summary/Conclusion

Motivations

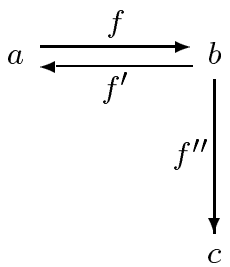
- Express Semantics
  Semantics Functor
- Express Compositions (or Structures) on (high-level) nets
  Limits, Colimits
- Properties of the composed net derived from properties of its components
  Liveness, T-Invariants, P-Invariants, ...

Graph

A *graph* G is given by:

1. G_0, a collection of nodes (or objects),
2. G_1, a collection of arrows (or morphims),
3. Two functions dom, cod

$$G_1 \underset{cod}{\overset{dom}{\rightrightarrows}} G_0$$

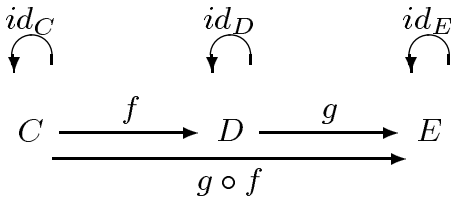


Each arrow $f \in G_1$ must have:
A source (domain) node a
A target (codomain) node b
It is denoted by $f : a \rightarrow b$ where
$dom(f) = a, cod(f) = b$.

Category

A category C is a graph with two additional functions:

1. $C_0 \xrightarrow{id} C_1$ the identity
   $c \mapsto id_c$
2. $C_2 \xrightarrow{\circ} C_1$ the composition
   $(g, f) \mapsto g \circ f$.



Verifiying the following properties:

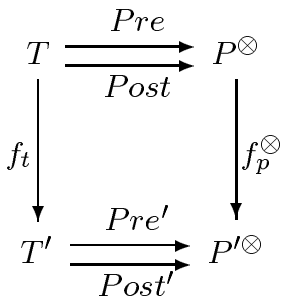
1. $\forall a \in C_0 \quad \Rightarrow dom(id_a) = cod(id_a) = a$
2. $\forall (g, f) \in C_2 \quad \Rightarrow dom(g \circ f) = dom(f), cod(g \circ f) = cod(g)$
3. $(g, f), (k, g) \in C_2 \quad \Rightarrow k \circ (g \circ f) = (k \circ g) \circ f$, associativity
4. $\forall f : a \to b \in C_1 \quad \Rightarrow f \circ id_a = id_b \circ f = f$, unit law

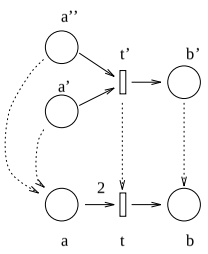
PetriG Category

- **PetriG** Category:

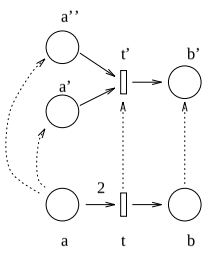
$$Ob(\mathbf{PetriG}) = \{PN =< T, P^{\otimes}, pre, post > \mid PN \text{ is a Petri Net}\}$$
$$Mor(\mathbf{PetriG}) = \{f =< f_t, f_p > \mid f \text{ is a PetriG-Morphism}\}$$

- A PetriG-Morphism is such that the following diagram commutes:





PetriG-Morphism



Not PetriG-Morphism

$P^{\otimes}$: free monoid , $f_p^{\otimes}$: extension of f_p to the a monoid homomorphism.
($f_p^{\otimes}$ respects generators !)

ANS and AHL Nets

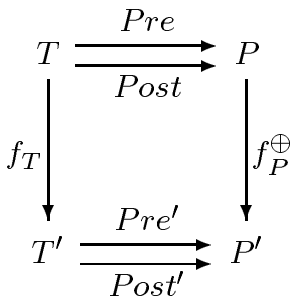
- Algebraic Net Specification (ANS),
  $ANS = (Spec, X, P, T, Pre, Post, sort)$ where:
  (1) $Spec = (S, F, EQ)$ an algebraic specification of abstract data type
  (2) X a S-sorted family of variables
  (3) P, T two sets of places, transitions
  (4) $Pre, Post : T \to [P \to T_{OP}(X)^{\oplus}]$
  (5) $sort : P \to S$

- Algebraic High-level net (AHL), $AHL = (ANS, A)$ where:
  (1) $ANS = (Spec, X, P, T, Pre, Post, sort)$ is an ANS
  (2) $A \in Alg(Spec)$ a $Spec$-algebra

AHL Category

- **AHL** Category:
  $Ob(\mathbf{AHL}) = \{AHL = (ANS, A) \mid AHL \text{ is an AHL net}\}$
  $Mor(\mathbf{AHL}) = \{f = (f_{ANS}, f_A) \mid f_{ANS} \text{ is a ANS-morphism, } f_A \text{ is a } Spec\text{-morph.}\}$

- A ANS-morphism, $f = (f_T, f_P) : ANS \to ANS'$
  (ANS, ANS') with the same $Spec$ is such that:
  (1) $f_T : T \to T', f_P : P \to P'$
  (2) $f_P^{\oplus}(pre(t)) = pre'(f_T(t))$
  $f_P^{\oplus}(post(t)) = post'(f_T(t))$
  $f_P^{\oplus} : [P \to T_{OP}(X)] \to [P' \to T_{OP}(X)]$
  $f_P^{\oplus}(M)(s) = \oplus_{h_P(p_1), p_1 \in P_1} M(p)$



- A $Spec$-morphism:
  $f_A : A_1 \to A_2$ is a family of maps:$\{f_{A_s} : A_{1_s} \to A_{2_s}\}_{s \in S} s.t.$
  $\forall f : s_1..s_n \to s, \forall a_1 \in A_{1_{s_1}}, \ldots a_n \in A_{1_{s_n}},$
  $f_{A_s}(f^{A_1}(a_1, \ldots, a_n)) = f^{A_2}(f_{A_{s_1}}(a_1), \ldots, f_{A_{s_n}}(a_n))$

Algebraic Specifications

- Algebraic Specification
  An algebraic specification is a triple: $Spec = (S, F, EQ)$ where:
  S is a set of sorts,
  F is a set of function symbols together with a mapping type $F : S^* \rightarrow S$
  $(f : s_1 \ldots s_n \rightarrow s)$
  EQ a set of equations of the form $t = t'$, with $t, t' \in (T_F)_s$

- *Spec*-Algebra
  $A = (\{A_s\}_{s \in S}, A_F)$ where
  $A_F = \{f^A : A_{s_1} \times \ldots \times A_{s_n} \rightarrow A_s \mid \exists f \in F, f : s_1..s_n \rightarrow s\}$

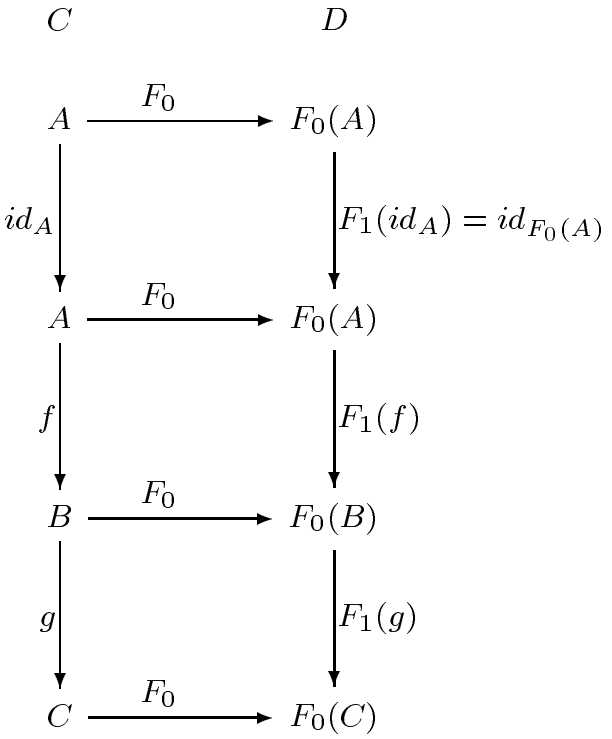
Functor

Let C, D be two categories, a *functor* $F : C \rightarrow D$ is a pair of functions:

1. $F_0 : C_0 \rightarrow D_0$
2. $F_1 : C_1 \rightarrow D_1$

such that:

1. $\forall f : A \rightarrow B \in C_1 \Rightarrow$
   $F_1(f) : F_0(A) \rightarrow F_0(B) \in D_1$
2. $\forall A \in C_0 \Rightarrow$
   $F_1(id_A) = id_{F_0(A)}$
3. $\forall g \circ f \in C_1 \Rightarrow$
   $F_1(g \circ f) = F_1(g) \circ F_1(f) \in D_1$



Semantics Functor: AHL $\xrightarrow{Sem}$ PetriG

Sem : **AHL** $\rightarrow$ **PetriG**

(1) $Sem(ANS, A) = (T_{PN}, P_{PN}, pre_{PN}, post_{PN})$ where:

- (a) $T_{PN} = \{(t, ass_A) \mid ass_A \in [Var(t) \rightarrow A], t \in T\}$
- (b) $P_{PN} = \cup_{p \in P} A_{sort(p)} \times p$
- (c) $pre_{PN}(t, ass_A) = ass_A^{\oplus}(pre(t))$
- (d) $post_{PN}(t, ass_A) = ass_A^{\oplus}(post(t))$

$ass_A^{\oplus} : T_{OP}(X)^{\oplus} \rightarrow (P \times A)^{\oplus}$

(2) $Sem((f_{ANS}, f_A)) = ... = (f_T, f_P)$

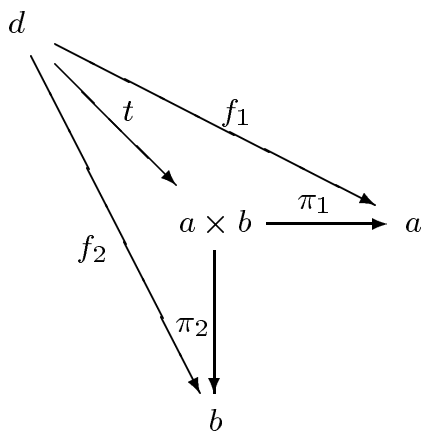
$$\begin{array}{ccc} (ANS, A) & \xrightarrow{Sem} & Sem(ANS, A) \\ \downarrow (f_{ANS}, f_A) & & \downarrow (f_T, f_P) \\ (ANS', A') & \xrightarrow[Sem]{} & Sem(ANS', A') \end{array}$$

Product

Given C a category, let $a, b \in Ob(C)$ then the **product** of a, b is an object $a \times b \in Ob(C)$ together with 2 arrows π_1, π_2 s.t.

- $\pi_1 : a \times b \rightarrow a$,$\pi_2 : a \times b \rightarrow b$
- $\forall d \in Ob(C)$ with $f_1 : d \rightarrow a$, $f_2 : d \rightarrow b$, $!\exists t : d \rightarrow a \times b$ s.t. $\pi_i \circ t = f_i$

In *Set*: Cartesian product

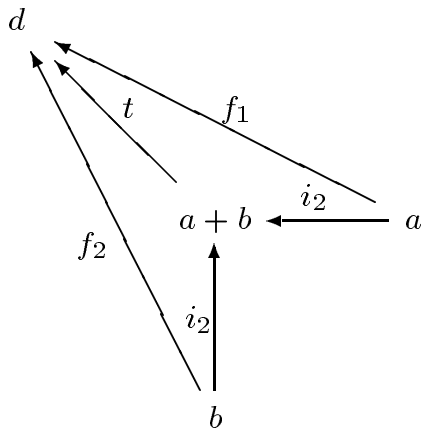


Coproduct

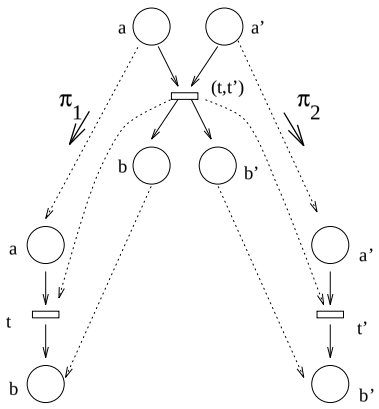
Given C a category, let $a, b \in Ob(C)$ then the **coproduct** of a, b is an object $a + b \in Ob(C)$ together with 2 arrows i_1, i_2 s.t.

- $i_1 : a \rightarrow a + b$,$i_2 : b \rightarrow a + b$
- $\forall d \in Ob(C)$ together with
  $f_1 : a \rightarrow d, f_2 : b \rightarrow d$
  $!\exists t : a + b \rightarrow d$ s.t. $t \circ i_i = f_i$

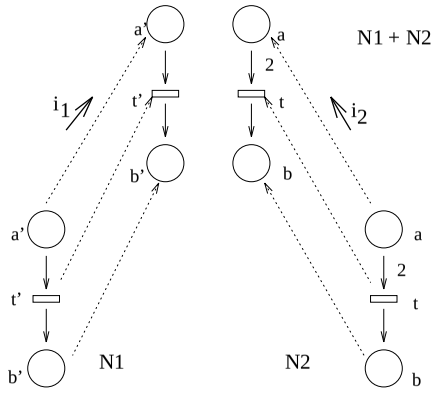
In *Set*: Disjoint Union



Product / Coproduct in PetriG



Product



CoProduct

Summary: How is used Category theory ?

- **Category**: Model of a paradigm
  Objects + Morphisms
- **Semantics**: Functor into another category
  Mapping between two categories: semantics functor
- **Properties**: Limits, Colimits
  (1) Prove that the Category is cocomplete (complete):
  $\Rightarrow \exists$ Colimits (Limits), esp. Pushout (Pullback)
  $\Rightarrow$ Composition of nets is possible
  (2) Prove Semantics functor preserves colimits:
  $\Rightarrow$ Semantics of Compound Objects is pushout of semantics of Components
- **Invariants**: Functor into another category
  (3) Prove that Invariant functor preserves limits, colimits
  $\Rightarrow$ Invariants of compound objects are derived from Invariants of components

Conclusion

- Category is the model of a given paradigm
  Category gives the structure
- Most important is the morphism:
  If morphism changes
  $\Rightarrow$ behavior of image changes
  $\Rightarrow$ cocompleteness changes
  $\Rightarrow$ colimits change
  $\Rightarrow$ properties change

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