Approximating the Tutte polynomial of a graph

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The Tutte polynomial of a graph G is a two-variable polynomial $T(G; x, y)$ that encodes many interesting properties of the graph. We study the complexity of the following problem, for rationals x and y — take as input a graph G, and output a value which is a good approximation to $T(G; x, y)$. Jaeger, Vertigan and Welsh have completely mapped the complexity of exactly computing the Tutte polynomial. They have shown that this is #P-hard, except along the hyperbola $(x-1)(y-1) = 1$ and at the four special points $(x, y) = (1, 1)$, $(0, -1)$, $(-1, 0)$, and $(-1, -1)$, where the value of the polynomial can be computed in polynomial time.

This talk describes joint work with Mark Jerrum, in which we aim to map the complexity of approximately computing the polynomial.

First I'll describe some older results, including a large region of intractability which includes the specialisation to the "flow polynomial" $F(G, \lambda)$ for $\lambda > 2$. This implies, for instance, that subject to the complexity-theoretic assumption RP $\neq$ NP, there is no fully-polynomial randomised approximation scheme (FPRAS) for counting nowhere-zero 6 flows, even though the corresponding decision problem is in P. Subject to the complexity-theoretic assumption, most of the region in which at least one of the parameters x and y is less than -1 is intractable.

Along a hyperbola $(x-1)(y-1) = q$, for an integer q, the Tutte polynomial is equivalent to the partition function of the q-state Potts model from statistical physics. The Ising model is just the specialisation of the q-state Potts model to the case $q = 2$, so Jerrum and Sinclair's FPRAS for the ferromagnetic Ising model provides an FPRAS for the Tutte polynomial along the positive branch of the hyperbola $q = 2$.

The talk will describe some recent complexity results giving evidence that the Tutte polynomial is difficult to approximate above this hyperbola. These results are subject to a weaker complexity-theoretic assumption, namely that there is no FPRAS for a certain logically-defined complexity class $\#RH\Pi_1$. I will describe this class and some of its complete problems, for example, the key problem of counting independent sets in a bipartite graph.

The proof that the Tutte polynomial is as hard to approximate as this class when $q > 2$ and $y > 1$ exploits the first order phase transition of the "random cluster" model, which is a probability distribution on graphs that is closely related to the q-state Potts model.